Perfectoid spaces

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1 Introduction and motivation: tilting

Fix a prime p. Everything we do today will be in relation to p.

Our goal is to construct a correspondence between certain objects in characteristic 0 and certain objects in characteristic p. Specifically, we will define perfectoid fields (which can have characteristic 0 or p), as well as perfectoid algebras and perfectoid spaces over them. We will construct tilting functors from perfectoid objects in characteristic 0 to those characteristic p, which (after fixing a perfectoid field and its tilt as base fields) will be equivalences of categories.

As an example, we like to think of $\mathbb{F}_p((t))$ as similar to $\mathbb{Q}_p$, where "t corresponds to p". One might hope that field extensions, algebras, schemes, and so on over one field would "correspond" to those over the other. This fails quickly: $\mathbb{F}_p((t))$ has infinitely many separable degree-p extensions, but $\mathbb{Q}_p$ has only $p+1$ of them (or 7 if $p=2$). The problem is essentially that $\mathbb{F}_p((t))$ has too many extensions with wild ramification. But if we adjoin p-power roots and pass to completions, then these excess ramified extensions go away, and the resulting fields $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle = \widehat{Q_p(p^{1/p^\infty})}$ and $\widehat{\mathbb{F}_p((t))(t^{1/p^\infty})}$ actually have equivalent categories of finite separable extensions, and in particular, isomorphic absolute Galois groups. (This is a theorem of Fontaine and Wintenberger from 1979. Note that the categories of finite étale extensions of the completed fields are canonically equivalent to those of the uncompleted fields, via the functor of completion.) This equivalence of categories is given by the tilting equivalence, which in a certain sense does turn t into p.

Today we will define this tilting equivalence and introduce perfectoid spaces, which form a category to which this type of equivalence generalizes naturally.

2 Perfectoid fields

We begin with perfectoid fields, and then move on to perfectoid algebras and perfectoid spaces.

*Notes for a talk given in Berkeley's number theory seminar, organized by Xinyi Yuan, Sug Woo Shin, and Ken Ribet. Main references: *Perfectoid spaces and their applications* (ICM 2014), *Perfectoid spaces: a survey* (CDM 2013), and *Perfectoid spaces* (IHES 2012), all by Peter Scholze.

Definition 2.1. *A* perfectoid field *is a complete topological field K whose topology is given by a non-archimedean norm $K \to \mathbb{R}_{\geq 0}$ with dense image, such that $|p| < 1$ and the Frobenius $\Phi : \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective.*

Notice that perfectoid fields can have characteristic 0 or p. In characteristic p, the second-last condition is vacuous, and the last says that K is perfect.

Example 2.2. *The p-adic completions of $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\mu_{p^\infty})$, and $\overline{\mathbb{Q}_p}$, and the t-adic completions of $\mathbb{F}_p((t))(t^{1/p^\infty})$ and $\overline{\mathbb{F}_p((t))}$ are perfectoid fields. The first two tilt to the fourth, and the third tilts to the last.*

Now we must define the tilting functor.

Definition 2.3. *The tilting functor $(\cdot)^\flat$, from perfectoid fields to characteristic-p perfectoid fields, is constructed as follows. For a given perfectoid field K, define $\mathcal{O}_K^\flat = \lim_{\leftarrow}(\mathcal{O}_K/p)$, where the maps are the Frobenius, and $K^\flat = \operatorname{Frac} \mathcal{O}_K^\flat$. (Give $\mathcal{O}_K^\flat$ the inverse limit topology, and given $K^\flat$ the topology such that $\mathcal{O}_{K^\flat}$ is an open subset of it.) Given a morphism $K \to L$ of perfectoid fields, we get an induced morphism $\mathcal{O}_K \to \mathcal{O}_L$, thus $\mathcal{O}_K/p \to \mathcal{O}_L/p$, thus $\lim_{\leftarrow} \mathcal{O}_K/p \to \lim_{\leftarrow} \mathcal{O}_L/p$, thus $K^\flat \to L^\flat$.*

This is the obvious thing to do if you think about it. We want to turn a characteristic-0 field into a perfect characteristic-p field, so at some point we want to mod out by p. Before doing that, we must make p non-invertible by passing to the ring of integers. Then $\mathcal{O}_K/p$ is a ring of characteristic p, but it is neither perfect nor a field, because it has a lot of nilpotents. There are two canonical ways to make an $\mathbb{F}_p$-algebra perfect: take the direct or inverse limit under the Frobenius map. The direct limit would kill all the nilpotents and yield something like the residue field, which is too small. So we instead take the inverse limit, which "lifts the nilpotents out of nilpotency".

Proposition 2.4. *The tilting functor has the following properties:*

1. *The tilt of any perfectoid field is a characteristic-p perfectoid field. The tilt of a characteristic-p field is (canonically isomorphic to) itself.*

2. *As multiplicative monoids, $\mathcal{O}_K^\flat = \lim_{\leftarrow} \mathcal{O}_K$, where the transition maps are the p-th power, and similarly $K^\flat = \lim_{\leftarrow} K$. This gives a multiplicative map $(\cdot)^\sharp : K^\flat \to K$, which induces an isomorphism of value groups.*

3. *For a suitably chosen pseudouniformizer $\varpi^\flat \in \mathcal{O}_K^\flat$ and $\varpi = (\varpi^\flat)^\sharp \in \mathcal{O}_K$, we have $\mathcal{O}_{K^\flat}/\varpi^\flat \cong \mathcal{O}_K/\varpi$. ("Suitably chosen" means that $|\varpi| \geq |p|$; "pseudouniformizer" means additionally that $0 < |\varpi| < 1$.)*

4. *For a given perfectoid field K, tilting gives an equivalence of categories $\mathrm{FEt}(K) \simeq \mathrm{FEt}(K^\flat)$ of finite étale extensions, preserving degrees of extensions. In particular, this implies $G_K \cong G_{K^\flat}$.*

5. *(Fargues-Fontaine) Every perfectoid field L of characteristic p has infinite many untilts in characteristic 0, and these are parametrized by a curve (the Fargues-Fontaine curve).*[1] *Equivalently, they are parametrized by certain principal ideals in the ring of Witt vectors $W(\mathcal{O}_L)$.*

For (2), we refer to the following diagram, where the top row has maps of multiplicative monoids and the bottom row has maps of rings:[2]

$$\begin{array}{ccccccccc}
 & & \lim_{\leftarrow} \mathcal{O}_K & \longrightarrow \cdots \longrightarrow & \mathcal{O}_K & \xrightarrow{(\cdot)^p} & \mathcal{O}_K & \xrightarrow{(\cdot)^p} & \mathcal{O}_K \\
 & & \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_K^\flat & = & \lim_{\leftarrow} \mathcal{O}_K/p & \longrightarrow \cdots \longrightarrow & \mathcal{O}_K/p & \xrightarrow{\Phi} & \mathcal{O}_K/p & \xrightarrow{\Phi} & \mathcal{O}_K/p
\end{array}$$

Observation: because of this isomorphism, we get a natural multiplicative map $(\cdot)^\sharp : \mathcal{O}_K^\flat \to \mathcal{O}_K$ via projection onto the last copy of $\mathcal{O}_K$. This is interesting, because it goes from something in characteristic p to something in characteristic 0. This is analogous to the Teichmüller lift $[\cdot] : \mathbb{F}_p \to \mathbb{Z}_p$, or more generally $[\cdot] : A \to W(A)$ for A a perfect $\mathbb{F}_p$-algebra. This can be defined as $x \mapsto \lim_{n\to\infty}(\overline{x})^{p^n}$ in the case of $\mathbb{F}_p \to \mathbb{Z}_p$, but more generally it must be $x \mapsto \lim_{n\to\infty}(\overline{x^{1/p^n}})^{p^n}$ to account for the nontrivial action of the Frobenius on the residue field.

Example 2.5. *Let's work through the tilting process from $\mathbb{Q}_p\langle p^{1/p^\infty}\rangle$ and $\mathbb{F}_p(\widehat{(t))(t^{1/p^\infty}})$. If $K = \mathbb{Q}_p\langle p^{1/p^\infty}\rangle$, then $\mathcal{O}_K = \mathbb{Z}_p\langle p^{1/p^\infty}\rangle$, and $\mathcal{O}_K/p = \mathbb{Z}_p\langle p^{1/p^\infty}\rangle/p \cong (\mathbb{Z}_p/p\,\mathbb{Z}_p)[p^{1/p^\infty}] \cong \mathbb{F}_p[t^{1/p^\infty}]/(t)$. The following diagram (of rings) tells us the Frobenius inverse limit of $\mathcal{O}_K/p$:*

$$\begin{array}{ccccccc}
 & \cdots \longrightarrow & \mathbb{Z}_p\langle p^{1/p^\infty}\rangle & \xrightarrow{x\mapsto x^p} & \mathbb{Z}_p\langle p^{1/p^\infty}\rangle & \xrightarrow{x\mapsto x^p} & \mathbb{Z}_p\langle p^{1/p^\infty}\rangle \\
 & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{F}_p\langle t^{1/p^\infty}\rangle = \lim_{\leftarrow} & \longrightarrow \cdots \longrightarrow & \mathbb{F}_p[t^{1/p^\infty}]/(t) & \xrightarrow{\Phi} & \mathbb{F}_p[t^{1/p^\infty}]/(t) & \xrightarrow{\Phi} & \mathbb{F}_p[t^{1/p^\infty}]/(t) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathbb{F}_p\langle u^{1/p^\infty}\rangle = \lim_{\leftarrow} & \longrightarrow \cdots \longrightarrow & \mathbb{F}_p[u^{1/p^\infty}]/(u^{p^2}) & \longrightarrow & \mathbb{F}_p[u^{1/p^\infty}]/(u^p) & \longrightarrow & \mathbb{F}_p[u^{1/p^\infty}]/(u)
\end{array}$$

(The vertical maps on the bottom right are given by "sending t to u^{p^i}", but the vertical map on the left sends t to u. The maps from the bottom inverse limit are just quotient maps,

[1]More precisely, the Fargues-Fontaine curve is a regular noetherian scheme X_L of dimension 1, of infinite type over $\mathbb{Q}_p$. Its closed points are identified with equivalence classes of pairs (K, ι) where K is a characteristic-0 perfectoid field and $\iota : L \to K^\flat$ expresses $K^\flat$ as a finite extension of L; two pairs (K, ι) and (K', ι') are equivalent if $K \cong K'$ and ι' differs from ι by a power of Frobenius. We call $[K^\flat : L]$ the degree of the point (K, ι). There are infinitely many degree-1 points.

[2]Proof of (2): I claim the vertical map on the left is an isomorphism of multiplicative monoids. To prove this, we will write down the inverse map. Suppose we are given an element of $\mathcal{O}_K^\flat$, which is concretely an inverse system $(x_i)_{i\geq 0}$ of elements of $\mathcal{O}_K/p$ with each $x_{i+1}^p = x_i$. We need to construct a lift $(\widetilde{x}_i)_i$ in $\lim_{\leftarrow} \mathcal{O}_K$. Note that this new system of elements must have the property that each $\widetilde{x}_i$ is a lift of x_i that is a p^n-th power in $\mathcal{O}_K$ for all n. In fact, this characterizes $\widetilde{x}_i$ uniquely. Concretely, one can show (exercise) that for fixed i, the sequence $(\overline{x}_{i+n})^{p^n}$ is Cauchy (for arbitrary lifts $\overline{x}_{i+n}$, independently of choice), and $\widetilde{x}_i$ is its limit. One easily sees that this is a two-sided inverse to the projection map. So as monoids, we have $\mathcal{O}_K^\flat = \lim_{\leftarrow} \mathcal{O}_K$, and it follows easily that $K^\flat = \lim_{\leftarrow} K$.

but the maps from the middle inverse limit are p^i-th root followed by quotient maps.) So $\mathcal{O}_K^\flat = \mathbb{F}_p\langle u^{1/p^\infty}\rangle \cong \mathbb{F}_p\langle t^{1/p^\infty}\rangle$, and $K^\flat$ is its fraction field, which is the t-adic completion of $\mathbb{F}_p((t))(t^{1/p^\infty})$.

Notice also that in the earlier diagram, the element $(p^{1/p^n})_n$ in the top row maps to the element $(t^{1/p^n})_n$ in the bottom row, so the element $t = (t^{1/p^n})_n$ has $t^\sharp = p$. This justifies our earlier claim that t somehow corresponds to p.

3 Perfectoid algebras and spaces

Fix a perfectoid field K. Fix some pseudouniformizer $\varpi \in \mathcal{O}_K$ with $|p| \leq |\varpi| < 1$. For convenience, assume that ϖ comes equipped with a compatible family of p^n-th roots, $(\varpi^{1/p^n})_n$, which defines an element $\varpi^\flat \in \mathcal{O}_{K^\flat}$ with $(\varpi^\flat)^\sharp = \varpi$. (Such a ϖ exists, and the definitions will turn out to be independent of the choice. See the remarks about the choice of ϖ and $\varpi^\flat$ in Lemma 3.4 of the main paper.)

Definition 3.1. *A* perfectoid K-algebra *is a Banach K-algebra*[3] *R such that the set R° of power-bounded elements is open and bounded, such that the Frobenius gives a surjection $R^\circ/\varpi \to R^\circ/\varpi$. A* perfectoid affinoid K-algebra *is a Huber pair (R, R^+) such that R is a perfectoid K-algebra.*

(Note that for perfectoid algebras, R is automatically Tate, since K itself has a topologically nilpotent unit.) Perfectoid affinoid K-algebras have a tilting equivalence, similar to the case of finite étale field extensions:

Theorem 3.2. *If K is a perfectoid field, there is a canonical equivalence between the category of perfectoid affinoid algebras over K and over $K^\flat$, with the following property: if (R, R^+) maps to $(R^\flat, R^{\flat +})$, then we get an isomorphism $R^{\flat +} \cong \lim_{\leftarrow x \mapsto x^p} R^+$ as multiplicative monoids, and $R^{\flat +}/\varpi^\flat \cong R^+/\varpi$ as rings. (I'm told there is also an explicit construction of the tilt, but I couldn't find a reference for it.)*

We defer the construction and the proof sketch until later, since it involves almost mathematics and the almost purity theorem. First, we look at the adic spaces arising from perfectoid affinoid algebras and their tilts:

Theorem 3.3. *Let (R, R^+) be a perfectoid affinoid K-algebra, and let $X = \mathrm{Spa}(R, R^+)$, with presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$. Let $(R^\flat, R^{\flat +})$ be the tilt, with $X^\flat$, $\mathcal{O}_{X^\flat}$, and $\mathcal{O}_{X^\flat}^+$ defined correspondingly. Then we have:*

1. *The presheaves $\mathcal{O}_X$ and $\mathcal{O}_{X^\flat}$ are sheaves. (So X and $X^\flat$ are honest adic spaces.)*
2. *There is a homeomorphism $X \to X^\flat$, identifying rational subsets, given by $x \mapsto x^\flat$ where $x^\flat$ is the valuation defined by $|f(x^\flat)| = |f^\sharp(x)|$.*

[3] A Banach K-algebra is an R-algebra with a sub-multiplicative norm $||\cdot||$, respecting scalar multiplication, that makes R a complete metric space. Our Banach algebras are assumed to be associative, commutative, and unital.

3. *For every rational subset $U \subset X$ with tilt $U^\flat \subset X^\flat$, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is a perfectoid affinoid K-algebra with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.*

4. *For all $i > 0$, the K°-module $H^i(X, \mathcal{O}_X^+)$ is annihilated by the maximal ideal $\mathfrak{m}$.*

Now we are ready to define perfectoid spaces and the category thereof.

Definition 3.4. *An* affinoid perfectoid space *over K is an adic space isomorphic to* $\mathrm{Spa}(R, R^+)$ *for (R, R^+) a perfectoid affinoid K-algebra. A* perfectoid space *over K is an adic space over* $\mathrm{Spa}(K, \mathcal{O}_K)$ *that is locally affinoid perfectoid. Morphisms between perfectoid spaces are the morphisms of adic spaces.*

The tilting equivalence glues from affinoid open sets to give an equivalence between categories of perfectoid spaces over K and over $K^\flat$. Equivalently, the tilt of a perfectoid space X can be defined as any perfectoid space $X^\flat$ such that $\mathrm{Hom}(\mathrm{Spa}(R^\flat, R^{\flat+}), X^\flat) = \mathrm{Hom}(\mathrm{Spa}(R, R^+), X)$ for all perfectoid affinoid K-algebras (R, R^+), functorially in (R, R^+). One can then show that the tilt of any perfectoid space exists and is unique up to unique isomorphism.

Example 3.5. *The perfectoid closed unit disk over K is defined as*

$$\mathrm{Spa}(K\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle, \mathcal{O}_K\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle).$$

The perfectoid $\mathbb{A}^n$, $(\mathbb{A}_K^n)^{perf}$, is an infinite union of disks of increasing radius; it is not affinoid, or even quasicompact. The perfectoid $\mathbb{P}^n$, $(\mathbb{P}_K^n)^{perf}$, is glued from $n+1$ copies of either affine space or the closed disk. These spaces can be realized as "approximate Frobenius inverse limits" of the corresponding non-perfectoid objects in the category of adic spaces. (The inverse limits don't actually exist in the category of adic spaces, but they exist in a weak sense, analogous to coarse moduli spaces.)

4 Idea of proofs of tilting

As before, fix a perfectoid field K, along with a pseudouniformizer ϖ dividing p and a pseudouniformizer $\varpi^\flat$ of $K^\flat$ such that $(\varpi^\flat)^\sharp = \varpi$.

To prove the tilting equivalence for finite étale extensions of K and $K^\flat$, we want to give an equivalence of categories:

$$K_{\text{fét}} \cong K^{\circ a}_{\text{fét}} \cong (K^{\circ a}/\varpi)_{\text{fét}} \cong (K^{\flat\circ a}/\varpi^\flat)_{\text{fét}} \cong K^{\flat\circ a}_{\text{fét}} \cong K^\flat_{\text{fét}}$$

The first and last categories consist of finite étale algebras over K and $K^\flat$. The rest are categories of almost algebras over various rings. The middle equivalence is the easiest, since the two rings in question are isomorphic, compatibly with the "almost structure". The outermost equivalences are harder, and amount to Faltings' almost purity theorem. The remaining two involve deformation theory in the almost setting, including almost cotangent complexes. Similarly, for perfectoid algebras, we want equivalences of categories:

$$K - \mathrm{Perf} \cong K^{\circ a} - \mathrm{Perf} \cong (K^{\circ a}/\varpi) - \mathrm{Perf} \cong (K^{\flat\circ a}/\varpi^\flat) - \mathrm{Perf} \cong K^{\flat\circ a} - \mathrm{Perf} \cong K^\flat - \mathrm{Perf}$$

These categories are as above, but with the words "finite étale" replaced by "perfectoid". In both cases, we need to define all but the leftmost and rightmost categories.

Recall that the tilting equivalence for perfectoid spaces follows from considerations on affinoid patches, so our goal for the rest of the talk is to define the categories above and explain the equivalences between them.

5 Almost mathematics

We want to define categories of "almost modules", "almost algebras", and so on, over the ring $K^\circ = \mathcal{O}_K$. The idea of almost mathematics is to treat modules annihilated by the maximal ideal $\mathfrak{m} = K^{\circ\circ}$ (i.e. vector spaces over the quotient field $K^\circ/K^{\circ\circ}$) as negligible. We call such modules *almost zero*.

More precisely, we want an abelian category $K^{\circ a}$−mod of "almost K°-modules", equipped with an exact functor $K^\circ - \text{mod} \to K^{\circ a} - \text{mod}$ that sends almost zero modules to the zero object. There is a natural construction of this form associated to any *thick* (or *weakly Serre*) subcategory of an abelian category; namely, a full subcategory T such that for any short exact sequence $0 \to M' \to M \to M'' \to 0$, M is in T if and only if M' and M'' are in T. In our case, one can easily show that if M is $\mathfrak{m}$-torsion, then M' and M'' are, and if M' and M'' are $\mathfrak{m}$-torsion, then M is $\mathfrak{m}^2$-torsion. But $\mathfrak{m}^2 = \mathfrak{m}$, because the value group is dense.

So we get a category of almost K°-modules, equipped with a functor $K^\circ - \text{mod} \to K^{\circ a} - \text{mod}$, which we denote $M \mapsto M^a$. This is an exact functor, and in fact it has left- and right-adjoints. We also have a $\otimes$-structure on $K^{\circ a}$ given by $M^a \otimes N^a = (M \otimes N)^a$, an internal Hom, a Hom-tensor adjunction, and so on. Note also that the functor $K^\circ - \text{mod} \to K - \text{mod}$ given by tensoring up to K factors through $K^{\circ a}$-mod, as it is (isomorphic to) the Serre quotient of K°-mod by the subcategory of all torsion modules. This will give the two outermost functors in the earlier chain of equivalences of categories, once we know about almost algebras. (Proving that they are equivalences is more difficult.)

The book of Gabber and Romero develops an extensive theory of almost commutative algebra and algebraic geometry; nearly every ring-theoretic notion imaginable can be systematically replicated within the almost category. We don't have time for this, so we omit the definitions of finite étale almost algebras and perfectoid almost algebras, and will be content with sketching the definition of almost algebras.

Recall that an ordinary algebra over some ring A is just an A-module M equipped with a multiplication map $M \otimes_A M \to M$ satisfying some properties. Since we have tensor products in the almost category, we can transport the same definition to the almost setting, giving a category of $K^{\circ a}$-algebras.